involve the combinations

$$\int_V (d\mathbf{x})(\vartheta^k+\partial_l f^{kl})=P^k+\int_S d\sigma_l f^{kl}$$

and

$$\int_V (d\mathbf{x})[x^k(\vartheta^l+\partial_m f^{ml})-x^l(\vartheta^k+\partial_m f^{mk})] = J^{kl}+\int_S d\sigma_m(x^k f^{ml}-x^l f^{mk}).$$

The asymptotic vanishing of these surface integrals is in the nature of a boundary condition characterizing a physically closed system. This property can be verified, if one retains only the slowly decreasing terms in the asymptotic behavior of the fields,

$$|\mathbf{x}|\to\infty:\quad q^{kl}\sim\delta_{kl}+(\kappa/4\pi)P^0\partial_k\partial_l|\mathbf{x}|\,,$$
$$\Pi_{kl}\sim-1/(8\pi)P_m[\delta_{lm}\partial_k|\mathbf{x}|^{-1}+\delta_{km}\partial_l|\mathbf{x}|^{-1}-\tfrac{1}{2}\delta_{kl}\partial_m|\mathbf{x}|^{-1}-\tfrac{3}{4}\partial_k\partial_l\partial_m|\mathbf{x}|].$$

The outcome of these considerations is the commutation properties

$$-i[P^0,J^{0k}]=P^k\,,$$
$$-i[J^{0k},J^{0l}]=-J^{kl}\,,$$

which completes the formal verification of Lorentz invariance. But a much more careful examination will be required to test whether the loosely stated physical boundary conditions can be maintained as assertions about operators in relation to a class of physical states.

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# Dissipative Potentials and the Motion of a Classical Charge. II

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In an earlier paper by the author, examples of the motion of a point charge were found to be consistent with the hypothesis of Abraham that the mass of an electron (or positron) is entirely electromagnetic. Further consequences of this hypothesis are developed. It is shown that the conservation laws of the electromagnetic field and Maxwell's equations require that the total Lorentz force (including the self-force) on the charge should vanish. This result can be expressed as a Lagrangian equation of motion. The canonical four momentum of the charge is the product of the magnitude of the charge by the four potential of the field at the position of the charge. When the dissipative form of the potential for an unconfined point charge is used, the integro-differential equation of motion of the earlier paper is obtained for a particle with zero "bare" mass. A mechanical momentum and mass are defined; these are associated with the singular part of the Green's function for the D'Alembert equation. The rate of change of this mechanical momentum is equal to the sum of the external force, the radiation damping force (with the correct sign obtained by the use of the retarded fields), and the gradient at the position of the charge of its Coulombic self-potential energy. For a particle assumed to follow a continuous trajectory, the integrals in the integro-differential equation of motion are evaluated by a procedure in agreement with, but much simpler than, that of Dirac. The result is the unrenormalized equation of Dirac for a particle whose mass is the divergent Coulombic self-energy. The effective momentum and mass in this equation are reduced to half of the mechanical momentum and mass by the force term arising from the gradient of the Coulombic self-potential energy.

## INTRODUCTION

In a previous paper, I[1], an integro-differential equation for the motion of a point charge was described and applied to the examples of motion of a free particle and of a nonrelativistic simple harmonic oscillator. The equation was obtained by assuming the validity of the Lorentz force equation in addition to Maxwell's field equations. The force on the charge at the field point was taken to be the Lorentz force produced by the fields of a source charge in the limit where the field charge is identified with the source. It was pointed out that the motion of the charge in the examples considered was consistent with the Abraham hypothesis that the mass of the electron (or positron) is wholly electromagnetic. In the present paper further consequences of this hypothesis are developed. It is shown in Sec. 1 that the conservation laws of the electromagnetic field and Maxwell's equations require that the total Lorentz force (including the self-force) on a point charge vanish. In Sec. 2, it is shown that this result can be derived from a Lagrangian function, similar to the usual Lagrangian for a particle in an electromagnetic field, but with the bare mass suppressed. The canonical momentum of the charge obtained from this Lagrangian is $p_\sigma=eA_\sigma(z)$ where $A_\sigma$ is the four potential of the field at the position $z$ of the charge $e$. When the dissipative form (3.1) of the potential for an unconfined point charge, plus the potential of the external fields, is used for $A_\sigma$, the integro-

[1] B. Leaf, Phys. Rev. **127**, 1369 (1962). Referred to as I in this paper.

differential equation of I (with the bare mass suppressed) is again obtained. Accordingly, this equation is derivable from Maxwell's equations and the conservation laws for the electromagnetic field.

In Sec. 3, the Fourier transforms appearing in the dissipative potentials are identified with the singular and nonsingular parts of the Green's function[2] for the D'Alembert equation. The mechanical momentum and mass of the charge are associated with the singular part of the Green's function, and an equation is formulated for the rate of change of mechanical momentum (3.13). In addition to the external and damping forces appearing in this equation, there is a force arising from the gradient at the position of the charge, in its instantaneous rest frame, of the Coulombic self-potential energy of the charge.

In Sec. 4, the remaining integrals in the equations are evaluated for the case that the charge is assumed to follow a continuous trajectory. The method of evaluation as described in the Appendix is in agreement with the procedure of Dirac[3] but greatly simplified. In Sec. 5, a discussion of results is given. The integrodifferential equation of I reduces for continuous trajectories to the unrenormalized Dirac equation. The effective momentum and mass in this equation, however, are reduced to half of the mechanical momentum and mass by the force term arising from the gradient of the Coulombic self-potential energy. (This is the usual factor $\frac{1}{2}$ in the self-energy of a charge appearing when the charge is assembled from its elements.) The mass is the divergent Coulombic self-energy. The correct sign for the radiation damping is given by the use of the retarded fields of the charge, in agreement with the causality principle discussed in I.

## 1. THE CONSERVATION LAWS AND THE EQUATION OF MOTION

On the basis of a field theory, the conservation laws may be expressed in the form

$$\partial T_{\mu\nu}/\partial x_\nu = 0\,, \tag{1.1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the field. (The notation is suitable to the use of a pseudo-Euclidean metric in special relativity with the speed of light, $c=1$.) We consider the consequences of the assumption that the electromagnetic tensor,

$$T_{\mu\nu} = (1/4\pi)[F_{\mu\sigma}F_{\nu\sigma} - \tfrac{1}{4}F_{\sigma\sigma}{}^2\delta_{\mu\nu}]\,, \tag{1.2}$$

when introduced into (1.1) completely accounts for the dynamics of a point singularity in the field, which is the model which we adopt for a classical charge. According to this model, the current density associated with the charge is[4]

$$j_\sigma(x) = \int_{-\infty}^{\infty} d\tau e\dot{z}_\sigma \delta^4(x-z)\,, \tag{1.3}$$

where $z=x(\tau)$ is the position of the charge $e$ at proper time $\tau$, and $\dot{z}$ is its four velocity.

As is well known[5]

$$\partial T_{\mu\nu}/\partial x_\nu = -F_{\mu\nu}j_\nu\,, \tag{1.4}$$

so that (1.1) implies that

$$F_{\mu\nu}j_\nu = \int d\tau e F_{\mu\nu}(x)\dot{z}_\nu \delta^4(x-z) = 0\,. \tag{1.5}$$

Integration over a three-dimensional hypersurface normal to the world line and enclosing the charge gives the equation of motion of the charge:

$$F_\mu(z) \equiv \int \cdots \int d^4x \delta^4(x-z) e F_{\mu\nu}(x)\dot{z}_\nu = eF_{\mu\nu}(z)\dot{z}_\nu = 0\,. \tag{1.6}$$

Equation (1.6) states that the Lorentz force $F_\mu(z)$ on the particle vanishes. In this equation, as in (1.2) the fields $F_{\mu\nu}$ are the total fields including the self-field of the charge. This equation is equivalent to the equation of motion assumed in I (2.3) or I (5.1), in which the bare mass has been suppressed.

Dirac[3] has expressed the conservation laws in the form

$$\oint_\Sigma T_{\mu\nu} dS_\nu = B_\mu(\tau) - B_\mu(\tau_0)\,, \tag{1.7}$$

where the integral is taken over any tube surrounding the world line of the charge from $\tau_0$ to $\tau$. The flow of energy or momentum out from the surface of the tube must depend only on conditions at the two ends of the length of tube. Using Gauss' theorem to transform the surface integral into an integral over the enclosed volume, we find

$$\int_{\tau_0}^{\tau} d\tau \int\int\int d\mathbf{x}^0 \partial T_{\mu\nu}/\partial x_\nu = \int_{\tau_0}^{\tau} d\tau \dot{B}_\mu(\tau)\,, \tag{1.8}$$

where $d\mathbf{x}^0$ is the element of hypersurface normal to the world line, so that

$$\int\int\int d\mathbf{x}^0 \partial T_{\mu\nu}/\partial x_\nu = -eF_{\mu\nu}(z)\dot{z}_\nu = \dot{B}_\mu(\tau)\,. \tag{1.9}$$

We see that (1.1) or (1.6) require $\dot{B}_\mu(\tau)=0$, and not

$$\dot{B}_\mu(\tau) = -[m_{\text{eff}} - e^2/2|\xi|]\ddot{z}_\mu \quad (\xi \to 0)\,, \tag{1.10}$$

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[2] D. Ivanenko and A. Sokolow, *Klassische Feldtheorie* (Akademie-Verlag, Berlin, 1953), p. 58.

[3] P. A. M. Dirac, Proc. Roy. Soc. (London) **A167**, 148 (1938).

[4] F. Rohrlich, *Lectures in Theoretical Physics* (Interscience Publishers, Inc., New York, 1960), Vol. II, p. 241.

[5] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1951), p. 88.

as assumed by Dirac in his renormalization procedure. To choose any value other than zero for $\dot{B}_\mu$ is equivalent to assuming that $T_{\mu\nu}$ in (1.1) consists of some tensor of nonelectromagnetic origin[6] in addition to the electromagnetic tensor (1.2). This is contrary to the Abraham hypothesis.

## 2. POTENTIALS OF FIELD AND CANONICAL MOMENTA

An alternative form of the equation of motion (1.6) can be obtained in terms of the potentials of the field. Since

$$F_{\mu\nu}(x)=\partial A_\nu(x)/\partial x_\mu-\partial A_\mu(x)/\partial x_\nu, \tag{2.1}$$

therefore, according to (1.6),

$$e\dot{z}_\nu[\partial A_\nu(z)/\partial z_\mu-\partial A_\mu(z)/\partial z_\nu]=0. \tag{2.2}$$

But

$$dA_\mu(z)/d\tau=\dot{z}_\nu\partial A_\mu(z)/\partial z_\nu, \tag{2.3}$$

so that

$$deA_\mu(z)/d\tau=\dot{z}_\nu\partial eA_\nu(z)/\partial z_\mu. \tag{2.4}$$

In these equations, the potential at the position of the particle is[7]

$$A_\mu(z)=\int dx\delta^4(x-z)A_\mu(x). \tag{2.5}$$

In (2.4) we take $\partial/\partial z_\mu$ to be the partial derivative for fixed values of $\dot{z}$ and $\tau$. Accordingly,

$$deA_\mu(z)/d\tau=\partial[eA_\nu(z)\dot{z}_\nu]/\partial z. \tag{2.6}$$

This is the Lagrangian equation of motion resulting from the Lagrangian function,

$$L=eA_\nu(z)\dot{z}_\nu. \tag{2.7}$$

The canonical momentum, accordingly, is given by

$$p_\mu=\partial L/\partial\dot{z}_\mu=eA_\mu(z), \tag{2.8}$$

so that (2.4) becomes

$$dp_\mu/d\tau=\dot{z}_\nu\partial eA_\nu(z)/\partial z_\mu. \tag{2.9}$$

The right-hand side of (2.9) represents the four gradient, at the position of the charge at time $\tau$, of the scalar potential of the field in the instantaneous rest frame of the charge. The momentum defined in (2.8) agrees with the usual definition

$$p_\sigma=\mu\ddot{x}_\sigma+eA_\sigma, \tag{2.10}$$

where the bare mass $\mu$ is suppressed.

The Hamiltonian $H=\dot{z}_\nu p_\nu-L\equiv0$. But, in fact, (2.8) and (2.9) results from letting the mass $\mu\to0$ in the Hamiltonian equations of motion which are derived from the usual Hamiltonian[8]

$$H'(z,p)=(1/2\mu)[p_\nu-eA_\nu(z)]^2. \tag{2.11}$$

[6] L. Infeld and P. R. Wallace, Phys. Rev. **57**, 797 (1940).

[7] It is clear that the procedure followed in I for obtaining the potentials and fields at the position of the charge, which required setting equal to zero a parameter $\boldsymbol{\alpha}$, is replaced here by integration over the $\delta$ function, $\delta^4(x-z)$.

[8] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1950), p. 224.

## 3. IDENTIFICATION OF MECHANICAL PROPERTIES OF THE CHARGE

In order that the momentum $p_\mu$ as defined in (2.8) be covariant, $A_\mu$ must be expressed in Lorentz gauge. For an unconfined charge moving along an arbitrary trajectory, the part of the potential arising from the charge is given by the dissipative potentials which, as shown in I, are

$$A_\mu(\mathbf{x},t)$$
$$=\mp(ei/8\pi^3)\int\cdots\int d\mathbf{k}d\omega k^{-1}[\delta_\mp(\omega-k)-\delta_\mp(\omega+k)]$$
$$\times\int dt'v_\mu(t')\exp i\{\mathbf{k}\cdot[\mathbf{x}-\mathbf{x}(t)]-\omega(t-t')\} \tag{3.1}$$

(upper sign, advanced; lower sign, retarded). Separation into principal-part and $\delta$-function terms gives, respectively,

$$A_\mu{}^K(x)=(e/4\pi^3)\mathcal{P}\int\cdots\int d^4kd\tau(k_\sigma k_\sigma)^{-1}\dot{z}_\mu \times\exp[ik_\nu(x_\nu-z_\nu)], \tag{3.2}$$

$$A_\mu{}^D(x)=\mp(ei/4\pi^2)\int\cdots\int d^4kd\tau\delta(k_\sigma k_\sigma)\epsilon(\omega)\dot{z}_\mu \times\exp[ik_\nu(x_\nu-z_\nu)], \tag{3.3}$$

with the four-vector wave number defined as $k_\nu=(\mathbf{k},\omega)$, $\epsilon(\omega)=\pm1$ for $\omega\gtrless0$, and $x=(\mathbf{x},t)$.

In terms of the singular function[9]

$$\bar{D}(x)=(1/2\pi)^4\mathcal{P}\int\cdots\int d^4k(k_\sigma k_\sigma)^{-1}\exp(ik_\nu x_\nu) =(1/4\pi)\delta(x^2), \tag{3.4}$$

and the Pauli-Jordan commutation function[9]

$$D(x)=i/(2\pi)^3\int\cdots\int d^4k\delta(k_\sigma k_\sigma)\epsilon(\omega)\exp(ik_\nu x_\nu) =(1/2\pi)\epsilon(t)\delta(x^2), \tag{3.5}$$

the potentials become, with the use of (1.3)

$$A_\mu{}^K(x)=4\pi e\int d\tau\dot{z}_\mu\bar{D}(x-z) =4\pi\int\cdots\int d^4x'j_\mu(x')\bar{D}(x-x'), \tag{3.6}$$

$$A_\mu{}^D(x)=\mp2\pi e\int d\tau\dot{z}_\mu D(x-z) =\mp2\pi\int\cdots\int d^4x'j_\mu(x')D(x-x'). \tag{3.7}$$

[9] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959), pp. 649, 653.

Since $\bar{D}(x)$ and $D(x)$ satisfy the differential equation,

$$\partial^2\bar{D}(x)/\partial x_\nu{}^2 = -\delta^4(x) \quad \text{and} \quad \partial^2 D(x)/\partial x_\nu{}^2 = 0; \quad (3.8)$$

therefore, $\bar{D}(x)$ is the singular part, and $\mp\frac{1}{2}D(x)$ is a nonsingular part of the Green's function for the D'Alembert equation.[2] $D(x)$ is determined by the initial and boundary conditions of the system, which in the present case of an unconfined charge is the radiation condition at infinity. The causality principle discussed in I determines the choice of sign for $\mp D(x)$; the lower sign corresponding to the retarded solution gives the correct sign for the radiation damping. It is readily verified that $A_\mu{}^K(x)$ and $A_\mu{}^D(x)$ satisfy

$$\partial^2 A_\mu{}^K(x)/\partial x_\nu{}^2 = -4\pi j_\mu(x), \quad \partial^2 A_\mu{}^D(x)/\partial x_\nu{}^2 = 0, \quad (3.9)$$

and that each separately satisfies the Lorentz gauge condition,

$$\partial A_\nu(x)/\partial x_\nu = 0. \quad (3.10)$$

In the presence of external fields, the total potential $A_\mu(x)$ is the sum of the dissipative potential produced by the charge, and the external potential $A_\mu{}^{\text{ext}}(x)$ which, like $A_\mu{}^D(x)$, satisfies the homogeneous wave equation. Accordingly, the total potential becomes

$$A_\mu(x) = A_\mu{}^K(x) + A_\mu{}^D(x) + A_\mu{}^{\text{ext}}(x). \quad (3.11)$$

Corresponding to the decomposition of the total potential given in (3.11), the total particle momentum defined in (2.8) becomes the sum,

$$p_\mu = p_\mu{}^K + p_\mu{}^D + p_\mu{}^{\text{ext}}. \quad (3.12)$$

We now adopt as representing the "mechanical" momentum of the particle, the kinetic term $p_\mu{}^K$, which is derived from the singular part of the Green's function for the D'Alembert equation,[2] i.e., the part independent of boundary and initial conditions. Justification for this choice of $p_\mu{}^K$ to represent the mechanical momentum will appear in Eq. (4.5). The equation of motion (1.6) or (2.9) can now be written as

$$dp_\mu{}^K/d\tau = \dot{z}_\nu \partial e A_\nu{}^K(z)/\partial z_\mu + F_\mu{}^D(z) + F_\mu{}^{\text{ext}}(z), \quad (3.13)$$

where the Lorentz forces $F_\mu{}^D(z)$ and $F_\mu{}^{\text{ext}}(z)$ are obtained from (1.6) by using the fields $F_{\mu\nu}{}^D(x)$ and $F_{\mu\nu}{}^{\text{ext}}(x)$ derived, respectively, from the potentials $A_\mu{}^D(x)$ and $A_\mu{}^{\text{ext}}(x)$ which are solutions of the homogeneous wave equation.

## 4. EVALUATION OF INTEGRALS

According to Eqs. (3.4) to (3.7),

$$A_\mu{}^K(x) = e\int d\tau \dot{z}_\mu \delta(x-z)^2, \quad (4.1)$$

$$A_\mu{}^D(x) = \mp e\int d\tau \dot{z}_\mu \delta(x-z)^2 \epsilon[t-t(\tau)], \quad (4.2)$$

where $z=[\mathbf{x}(\tau), t(\tau)]$, so that with $z'=x(\tau')$,

$$p_\mu{}^K = e^2\int d\tau' \dot{z}_\mu' \delta(z-z')^2, \quad (4.3)$$

$$p_\mu{}^D = \mp e^2\int d\tau' \dot{z}_\mu' \delta(z-z')^2 \epsilon(\tau-\tau'). \quad (4.4)$$

As shown in the Appendix,

$$p_\mu{}^K = m\dot{z}_\mu, \quad (4.5)$$

$$p_\mu{}^D = \pm e^2 \ddot{z}_\mu, \quad (4.6)$$

$$\partial e A_\nu{}^K(z)/\partial z_\mu = -m(\dot{z}_\mu \ddot{z}_\nu + \tfrac{1}{2}\ddot{z}_\mu \dot{z}_\nu), \quad (4.7)$$

$$\partial e A_\nu{}^D(z)/\partial z_\mu = \mp e^2(\dot{z}_\mu d\ddot{z}_\nu/dt + \ddot{z}_\mu \ddot{z}_\nu + \tfrac{1}{3}\dot{z}_\nu d\ddot{z}_\mu/dt + \tfrac{1}{3}\dot{z}_\mu \dot{z}_\nu \dot{z}_\sigma d\ddot{z}_\sigma/dt), \quad (4.8)$$

where $m$ is the divergent quantity,

$$m = e^2\int d\xi \delta(\xi^2) = e^2/|\xi|, \quad \text{for} \quad \xi = 0. \quad (4.9)$$

Also, (4.7) and (4.8) give

$$\dot{z}_\nu \partial e A_\nu{}^K(z)/\partial z_\mu = \tfrac{1}{2}m\ddot{z}_\mu, \quad (4.10)$$

$$\dot{z}_\nu \partial e A_\nu{}^D(z)/\partial z_\mu = \pm(2e^2/3)(\tfrac{1}{2}d\ddot{z}_\mu/dt + \ddot{z}^2\dot{z}_\mu), \quad (4.11)$$

so that

$$F_\mu{}^K(z) = -dp_\mu{}^K/d\tau + \dot{z}_\nu \partial e A_\nu{}^K(z)/\partial z_\mu = -\tfrac{1}{2}m\ddot{z}_\mu, \quad (4.12)$$

$$F_\mu{}^D(z) = -dp_\mu{}^D/d\tau + \dot{z}_\nu \partial e A_\nu{}^D(z)/\partial z_\mu = \mp(2e^2/3)(d\ddot{z}_\mu/dt - \ddot{z}_\sigma{}^2\dot{z}). \quad (4.13)$$

Equation (3.13) becomes

$$dp_\mu{}^K/d\tau = \tfrac{1}{2}m\ddot{z}_\mu \mp (2e^2/3)(d\ddot{z}_\mu/dt - \ddot{z}_\sigma{}^2\dot{z}_\mu) + F_\mu{}^{\text{ext}}(z). \quad (4.14)$$

Equation (2.9) becomes

$$dp_\mu/d\tau = \tfrac{1}{2}m\ddot{z}_\mu \pm (2e^2/3)(\tfrac{1}{2}d\ddot{z}_\mu/dt + \ddot{z}_\sigma{}^2\dot{z}_\mu) + \dot{z}_\nu \partial e A_\nu{}^{\text{ext}}(z)/\partial z_\mu. \quad (4.15)$$

Equation (1.6) becomes

$$\tfrac{1}{2}m\ddot{z}_\mu = \mp(2e^2/3)(d\ddot{z}_\mu/dt - \ddot{z}_\sigma{}^2\dot{z}_\mu) + F_\mu{}^{\text{ext}}(z). \quad (4.16)$$

## 5. DISCUSSION

Equation (4.16) is the unrenormalized Dirac equation obtained when $\dot{B}_\mu(\tau)$ in (1.9) is taken to be zero, in accordance with the Abraham hypothesis. The integro-differential equation given in I (2.8) or I (5.1) is obtained when the dissipative potentials (3.2) and (3.3) are used to evaluate the self-forces $F_\mu{}^K(z)$ and $F_\mu{}^D(z)$ in (1.6). When the integrals are then evaluated for continuous trajectories as described in Sec. 4. the Dirac equation (4.16) results.

The evaluation of $p_\mu{}^K$ in (4.5) can be taken as justifying the choice of $p_\mu{}^K$ to represent the mechanical momentum of the particle. In this intepretation

$$m(\tau)=-\dot{z}_\mu p_\mu{}^K$$
$$=-(e^2/4\pi^3)\dot{z}_\mu\int d^4x\delta^4(x-z)$$
$$\times\mathcal{P}\int\cdots\int d^4k d\tau'(k_\sigma k_\sigma)^{-1}\dot{z}_\mu{}'\exp[ik_\nu(x_\nu-z_\nu')],\qquad(5.1)$$

is the rest mass of the particle. When evaluated for continuous trajectories,[10] according to (4.9), it is the divergent value in the rest frame of the particle at time $\tau$ of the Coulomb potential energy of the charge, or more correctly, twice the Coulomb energy. The usual explanation of the reduction of the Coulomb self-energy to half of $e\phi_0{}^K$ (where $\phi_0{}^K$ is the scalar self-potential in the rest frame of the charge) employs the model of a distributed charge $e$, where the factor $\frac{1}{2}$ appears in the work of creating the charge from its elements. Such an explanation is not suitable to our model of a point-like elementary charge $e$. A different explanation appears in the equation of motion (4.14). On the right-hand side we see that a part of the force changing the mechanical momentum of the particle is the term

$$\tfrac{1}{2}m\ddot{z}_\mu=\dot{z}_\nu\partial eA_\nu{}^K(z)/\partial z_\mu=-\partial e\phi_0{}^K/\partial z_\mu,\qquad(5.2)$$

which is the four gradient of the Coulomb self-potential energy.[11] As a result of this force, the inertial reaction of the particle to the action of external and damping forces, $F_\mu{}^{\text{ext}}$ and $F_\mu{}^D$ is only $\frac{1}{2}m\ddot{z}_\mu$ in (4.16). The effective mechanical momentum and mass in (4.16) are, therefore, $\frac{1}{2}p_\mu{}^K$ and $\frac{1}{2}m$. It should be noted that while $dp_\mu{}^K/d\tau=m\ddot{z}_\mu$, the rate of change of mechanical momentum as given in (4.5), depends on the use of the Lorentz gauge for $A_\mu{}^K(x)$ in (3.11), the self-force term of (4.12) or the inertial reaction of (4.16),

$$-F_\mu{}^K(z)=\tfrac{1}{2}m\ddot{z}_\mu,\qquad(5.3)$$

is independent of the gauge of $A_\mu{}^K(x)$.

Another point to be noted is that quantities like $A_\mu(z)$ which are functions of $z$, and were so considered in the Lagrangian $L(z,\dot{z})$ of (2.7), appear upon evaluation of integrals to depend on $\dot{z}$, $\ddot{z}$, etc. and not on $z$ at all. The Lagrangian formalism described earlier requires that these quantities be treated as functions of $z$.

In (4.16) we see again that the correct sign for the radiation damping term, $F_\mu{}^D=\mp(2e^2/3)(d\ddot{z}_\mu/dt-\ddot{z}_\sigma{}^2\dot{z}_\mu)$, is the lower sign, corresponding to the use of retarded fields, in agi eement with the causality principle discussed in I.

Instead of $\dot{B}_\mu(\tau)=0$ in (1.9) as required by the Abraham hypothesis, Dirac's renormalization procedure assumes

$$\dot{B}_\mu(\tau)=-m_{\text{eff}}\ddot{z}_\mu+\tfrac{1}{2}m\ddot{z}_\mu,\qquad(5.4)$$

according to (1.10). A negatively infinite mass is introduced to subtract the divergent positive electromagnetic mass. In effect, Dirac replaces $\frac{1}{2}m\ddot{z}_\mu$ on the left-hand side of (4.16) with $m_{\text{eff}}\ddot{z}_\mu$, where $m_{\text{eff}}$ is assumed to be the finite experimental mass. This procedure disposes of the factor $\frac{1}{2}$ in the inertial reaction of (4.16) as well as of the divergence. But Dirac's effective mass, $m_{\text{eff}}$ (or the negatively infinite term), is not calculated by the theory, whereas $m$ is given by (5.1). The fact that (5.1) diverges when evaluated for continuous[10] trajectories must be considered a defect of present theory.

## APPENDIX

While the integrals (4.5) to (4.8) can be evaluated by the procedure of Dirac,[3] the method is long and tedious. An alternative will be described here. Note first that the usual expression for evaluating the integral

$$\int dx\delta[f(x)]g(x)=\textstyle\sum_i g(x_i)/|f'(x_i)|,\qquad(\text{A1})$$

where the summation $\sum_i$ extends over the zeros of $f(x)$ (the values of $x=x_i$ for which $f(x_i)=0$), does not hold unless all the zeros are of first order. If $x_j$ is a zero of order $n$, then $f'(x_j)=f''(x_j)=\cdots=f^{(n-1)}(x_j)=0$, but $f^{(n)}(x_j)\neq0$. The same method used to derive (A1)[12] can be used to obtain, with $\xi=x-x_i$,

$$\int dx\delta[f(x)]g(x)=\sum_i\frac{n_i!}{|f^{(n_i)}(x_i)|}\sum_{k=0}^{\infty}\frac{g^{(k)}(x_i)}{k!}\int d\xi\delta(\xi^{n_i})\xi^k,\qquad(\text{A2})$$

where $n_i$ is the order of the zero of $f(x)$ at $x=x_i$. Since $\delta(\xi^{n_i})$ is an even function of $\xi$ for any value of $n_i$, therefore, only even value of the integer $k$ will contribute to the summation.

For the case in which $f(x)$ has a single zero of second order ($n=2$) at $x=x_j$, (A2) gives

$$\int dx\delta[f(x)]g(x)=\frac{2}{|f''(x_j)|}\sum_{k=0}^{\infty}\frac{g^{(k)}(x_j)}{k!}\int d\xi\delta(\xi^2)\xi^k\qquad(\text{A3})$$

for even-integral values of $k$. Also, with $\epsilon(x-x_j)$ defined as the step function of (3.3),

$$\int dx\delta[f(x)]g(x)\epsilon(x-x_j)=\frac{2}{|f''(x_j)|}\sum_{k=1}^{\infty}\frac{g^{(k)}(x_j)}{k!}\int d\xi\delta(\xi^2)|\xi|^k\qquad(\text{A4})$$

for odd-integral values of $k$. But since[12]

$$|\xi|\delta(\xi^2)=\delta(\xi),\qquad(\text{A5})$$

[10] The integral (5.1) is independent of the particle velocity. It can be evaluated for trajectories which have jump discontinuities. A jump discontinuity $\Delta\mathbf{x}^0$ in the rest frame of the particle at time $\tau$ gives for $m(\tau)$ the finite value, $m=e^2/|\Delta\mathbf{x}^0|$. As $\Delta\mathbf{x}^0\to0$, continuity of the trajectory is restored, but also the divergence (4.9) reappears.

[11] This term appeared in I(3.3) and I(A2.8) as $-\lim_{\alpha_0\to0}(\partial/\partial\alpha_0)\times(e^2/\alpha_0)$ in the rest frame of the charge. It was there incorrectly equated to zero. We now see that the only effect of this term is to reduce the divergent electromagnetic mass in the equation of motion by half.

[12] D. Ivanenko and A. Sokolow, *Klassische Feldtheorie* (Akademie-Verlag, Berlin, 1953), p. 16.

therefore, (A3) and (A4) reduce to

$$\int dx\delta[f(x)]g(x)=[2g(x_j)/|f''(x_j)|]\int d\xi\delta(\xi^2)\,, \quad \text{(A6)}$$

$$\int dx\delta[f(x)]g(x)\epsilon(x-x_j)=2g'(x_j)/|f''(x_j)|\,. \quad \text{(A7)}$$

(A7) is a finite expression, but (A6) contains the divergence,

$$\int d\xi\delta(\xi^2)=\int d\xi\delta(\xi)/|\xi|=1/|\xi| \quad \text{for} \quad \xi=0\,. \quad \text{(A8)}$$

To evaluate the integrals (4.5) to (4.8) it must first be noted that $\delta(z-z')^2$ is satisfied at the values of $\tau'$ for which $z=x(\tau')$ lies on a light cone of $z=x(\tau)$. Since $z$ is a point on the world line of the charge, and the charge is assumed to move at speeds less than that of light, the only value of $\tau'$ for which the $\delta$ function is satisfied is $\tau'=\tau$. Letting

$$f(\tau')=(z-z')^2\,, \quad g(\tau')=\dot{z}_\mu'\,,$$

where $f(\tau')$ has a second-order zero at $\tau'=\tau$, and applying (A6) and (A7) we obtain directly (4.5) and (4.6).

To obtain (4.7) and (4.8), we write

$$\begin{aligned}\partial eA_\nu^K(z)/\partial z_\mu &= -e^2\int d\tau'\dot{z}_\nu'\partial[\delta(z-z')^2]/\partial z_\mu' \\ &= -e^2\int d\tau'\frac{(z_\mu'-z_\mu)\dot{z}_\nu'}{(z_\sigma'-z_\sigma)\dot{z}_\sigma'}\frac{\partial}{d\tau'}\delta(z-z')^2 \\ &= e^2\int d\tau'\delta(z-z')^2 g(\tau')\,, \quad \text{(A9)}\end{aligned}$$

and similarly,

$$\partial eA_\nu^D(z)/\partial z_\mu=\mp e^2\int d\tau'\delta(z-z')^2\epsilon(\tau-\tau')g(\tau')\,, \quad \text{(A10)}$$

where

$$g(\tau')=(\partial/\partial\tau')\{(z_\mu'-z_\mu)\dot{z}_\nu'/(z_\sigma'-z_\sigma)\dot{z}_\sigma'\}\,. \quad \text{(A11)}$$

Expanding $g(\tau')$ in a Taylor's series about the point $\tau'=\tau$ gives

$$\begin{aligned}g(\tau')=&-(\dot{z}_\mu\ddot{z}_\nu+\tfrac{1}{2}\ddot{z}_\mu\dot{z}_\nu)+(\dot{z}_\mu d\ddot{z}_\nu/dt+\ddot{z}_\mu\ddot{z}_\nu+\tfrac{1}{3}\dot{z}_\nu d\ddot{z}_\mu/dt \\ &+\tfrac{1}{3}\dot{z}_\mu\dot{z}_\nu\dot{z}_\sigma d\ddot{z}_\sigma/dt)(\tau'-\tau)+O(\tau'-\tau)^2\,. \quad \text{(A12)}\end{aligned}$$

Applying (A6) and (A7) to (A9) and (A10), we obtain directly (4.7) and (4.8).

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# Generalized Hartree-Fock Approximation for the Calculation of Collective States of a Finite Many-Particle System*

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A finite many-particle system can have collective states for which the off-diagonal matrix elements of certain one-particle operators are of the same order of magnitude as the diagonal elements. In such cases it is suggested that the random-phase approximation is in need of generalization. Examples are the uniform translational motion of any system and the rotational motion of deformed nuclei. The generalization is suggested after a review and critical analysis of the Hartree-Fock approximation. The model single-particle wave functions of the latter are replaced by wave functions in a space labeled both by the particle variables and by the quantum numbers of the collective motion. These generalized amplitudes are defined field-theoretically, and a self-consistent scheme for their calculation is obtained from the equations of motion. In addition to the self-consistent potential defined in the enlarged space, the energies of the excited states also turn out to be given by a natural self-consistency requirement. The new calculational scheme is first applied to a systematic restudy of the random-phase approximation where the self-consistency requirement on the energies has previously been overlooked. As a first characteristic application we obtain without "pushing" the total mass of a system in uniform translation, and a reinterpretation of the Hartree-Fock average field.

## I. INTRODUCTION AND REVIEW

OUR aim in this paper is to describe a new method for the study of certain types of collective motion characteristic of *finite* many-particle systems. The method is viewed most naturally as an extension of the Hartree-Fock approximation (HFA),[1] and we have dubbed it the generalized Hartree-Fock approximation (GHFA). Several of the most fruitful recent develop-

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[1] A recent reference from which the reader may begin to trace the literature is W. H. Adams, Phys. Rev. **127**, 1650 (1962).